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1999 J. Phys. A: Math. Gen. 32 5285

(http://iopscience.iop.org/0305-4470/32/28/308)

J. Phys. A: Math. Gen. **32** (1999) 5285–5298. Printed in the UK PII: S0305-4470(99)02296-9

Bethe ansatz for the three-layer Zamolodchikov model

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Received 5 March 1999

Abstract. This paper is a continuation of our previous work (Boos H E and Mangazeev V V 1999 *J. Phys. A: Math. Gen.* **32** 3041–54). We obtain two more functional relations for the eigenvalues of the transfer matrices for the $sl(3)$ chiral Potts model at $q^2 = -1$. This model, up to a modification of boundary conditions, is equivalent to the three-layer three-dimensional Zamolodchikov model. From these relations we derive the Bethe ansatz equations.

1. Introduction

One of the open problems in the theory of integrable statistical systems is to construct the Bethe ansatz technique for three-dimensional integrable lattice models. A construction of such a model is connected with the problem of solving tetrahedron equations [3, 4] which insure integrability of a three-dimensional model. These are a system of thousands of equations in the simplest nontrivial case. Hence, the problem of solving them is very difficult.

There are only a few known integrable three-dimensional models which are interesting from the physical point of view. The first nontrivial example of such a model was proposed by Zamolodchikov in 1980 in [1, 2]. The tetrahedron equations for the Zamolodchikov model were proved by Baxter in [5].

Bazhanov and Stroganov [6] observed that the Zamolodchikov model and the three-dimensional free-fermion model were 'weakly equivalent', i.e., the free energy of the Zamolodchikov model and the free-fermion model satisfied the same symmetry and inversion relations. The assumption that analytical properties of the free energy were also the same resulted in a coincidence of the free energy for the Zamolodchikov model and the free-fermion model. In 1986 Baxter [7] calculated the partition function and free energy for the Zamolodchikov model with some modification of boundary conditions for the case of the infinite cubic lattice and for the case of the lattice which is infinite in two dimensions and finite in the third one. His result was similar to the result by Bazhanov and Stroganov for the free-fermion model but not the same. Namely, the free energy for the Zamolodchikov model was made up of a sum of two parts. The first part coincided with the free energy of the free fermion model and had the usual analytical properties for two-dimensional models. The second part was expressed in terms of the Euler dilogarithm function and had the cut in the complex plane. Therefore, the assumption that the free energy for the Zamolodchikov model and the

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0305-4470/99/285285+14$30.00

free-fermion model had the same analytical properties was incorrect. However, the similarity of these results was remarkable. Later Baxter and Quispel in [17] tried to clarify this fact. Namely, they constructed the Hamiltonian for the two- and three-layer Zamolodchikov model. The two-layer case turned out to correspond to the two-dimensional free-fermion model. The Hamiltonian for the three-layer case contained cubic interaction terms and did not seem to be the Hamiltonian for the free-fermion model.

Another important step in the theory of the integrable three-dimensional models was performed by Baxter and Bazhanov in 1992. Namely, they observed [9] that the $sl(n)$ chiral Potts model at $q^{2N} = 1$ [14, 15] was equivalent to the n-layer three-dimensional model which turned out to be the N-state generalization of the Zamolodchikov model. It was also mentioned that as for the Zamolodchikov model this equivalence is valid only up to some modification of boundary conditions which should not effect the partition function in the thermodynamic limit. The partition function for the Baxter–Bazhanov model was calculated in their next paper [10]. The result appeared to be connected in a remarkably simple way with that for the Zamolodchikov model.

We hope that a development of the Bethe ansatz technique for the Zamolodchikov and Baxter–Bazhanov models could shed a new light on the problems discussed above. Since the n-layer case of the Zamolodchikov model is equivalent to the $sl(n)$ chiral Potts model at $q^2 = -1$, up to a modification of boundary conditions, we can try to construct a Bethe ansatz for the $sl(n)$ chiral Potts model.

The Bethe ansatz technique is usually applicable to the study of effects connected with the finite size of a lattice. Therefore, there is a good chance that it will be useful for an investigation of the finite size corrections and the excitations.

Our first step is to develop this programme for the three-layer case of the Zamolodchikov model with modified boundary conditions, i.e., for the $sl(3)$ chiral Potts model at $q^2 = -1$.

This work is a continuation of our previous paper [18] where some functional relations for the $sl(3)$ chiral Potts model at $q^2 = -1$ were derived and the nested Bethe ansatz was constructed in the particular case when the vertical rapidity parameters coincide. Unfortunately, we did not succeed in solving these functional relations. Our goal here is to derive other functional relations and to obtain from them the Bethe ansatz equations for the general case.

The paper is organized as follows. In section 2 we recall the basic formulations of the $sl(3)$ chiral Potts model and its correspondence to the modified three-layer Zamolodchikov model. In section 3 we fix the definitions of the transfer matrices and discuss some of their simple properties. In section 4 we give two functional relations for the eigenvalues of the transfer matrices. In section 5 we obtain the Bethe ansatz equations. In the last section we give a brief discussion of the obtained results and directions for further investigation. In the appendix we outline the basic steps of the proof of one of the functional relations.

2. Basic formulations

The basic formulation of the Zamolodchikov model and it's generalization, the Baxter–Bazhanov model, can be found in papers [1, 2, 9]. In the last paper it was observed that the Boltzmann weights for the $sl(n)$ chiral Potts model at $q^{2N} = 1$ were a product of the n more simple weights (see formulae (3.7)–(3.13) of [9]). Hence, the 'star' weight for $sl(n)$ chiral Potts model appeared to be a product of the n weight functions interpreted as the Boltzmann weights for some N-state three-dimensional model:

Each weight of the product in the RHS depends on the eight spins with N possible values. For the case $N = 2$ this model turned out to be just the Zamolodchikov model rewritten by Baxter in the 'interaction-round-cube' form [5]. As it was already mentioned in

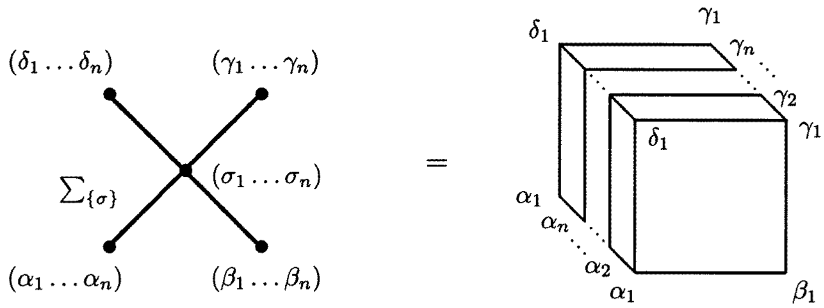

Figure 1.

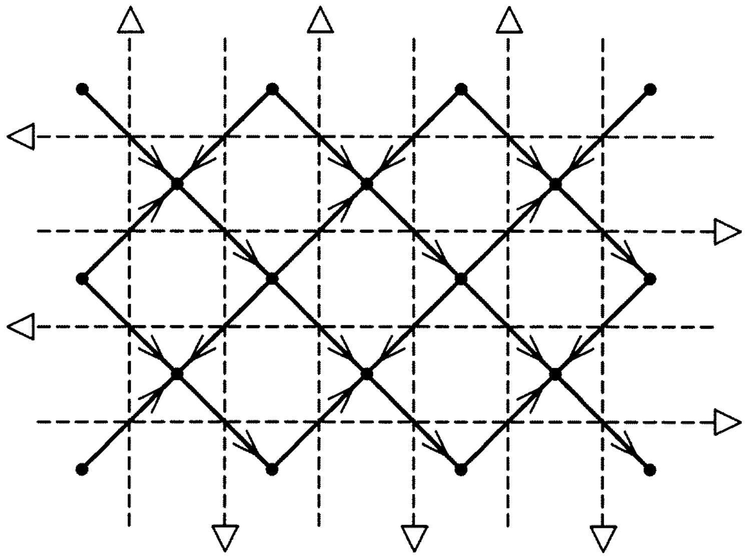

Figure 2.

the introduction, this equivalence is valid up to some modification of boundary conditions.

Since we study the three-layer case of the Zamolodchikov model we need to consider the $sl(3)$ chiral Potts model at $q^2 = -1$. The basic notations of this model were adduced in papers [14, 16] or in [9]. It can also be found in our previous paper [18], but to be independent we give some necessary basic definitions below.

The model is formulated on the square lattice (see figure 2).

The interaction is defined by two kinds of the Boltzmann weights $\bar{W}_{pq}(\alpha, \beta)$ and $(\bar{W}_{qp}(\alpha, \beta))^{-1}$ which depend on the neighbouring spin variables and spectral parameters. The rule for choosing these weights is shown in figure 3:

The Boltzmann weights depend on the rapidity parameters. Each rapidity variable is represented by three two-vectors $(h_i^+(p), h_i^-(p))$, $i = 1, 2, 3$ which specify the point p of the algebraic curve Γ defined by relations

$$\begin{pmatrix} h_i^+(p)^2 \\ h_i^-(p)^2 \end{pmatrix} = K_{ij} \begin{pmatrix} h_j^+(p)^2 \\ h_j^-(p)^2 \end{pmatrix} \qquad \forall i, j = 1, 2, 3, \tag{2.1}$$

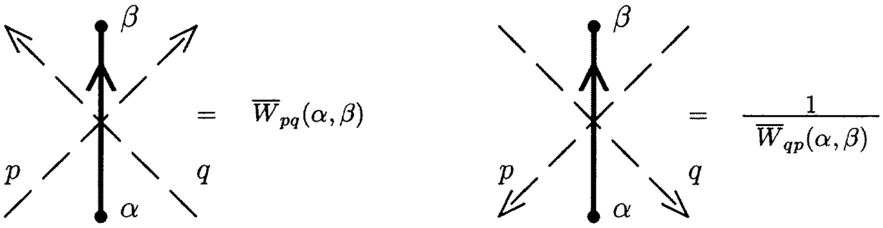

Figure 3.

where K_{ij} are 2×2 complex matrices of moduli satisfying

$$\det K_{ij} = 1 \qquad K_{ii} = K_{ij} K_{jk} K_{ki} = 1 \tag{2.2}$$

and indices i, j, k take values 1, 2, 3 *modulo* 3.

Further, we need the automorphism τ on the curve Γ defined as follows:

$$h_j^+(\tau(p)) = h_j^+(p) \qquad h_j^-(\tau(p)) = -h_j^-(p) \qquad j = 1, 2, 3. \tag{2.3}$$

The curve Γ can be defined in a different way which is also useful. Namely, for two arbitrary points p and q on this curve the following combination:

$$\Delta_{pq} = h_i^+(p)^2 h_i^-(q)^2 - h_i^-(p)^2 h_i^+(q)^2 \tag{2.4}$$

should be the same for all $i = 1, 2, 3$. It is easy to see that both these definitions are equivalent to each other.

The Boltzmann weights depend also on spin variables. Each spin variable is described by a two-vector

$$\alpha \equiv (\alpha_1, \alpha_2) \qquad \alpha_i \in Z_2 \qquad i = 1, 2. \tag{2.5}$$

Then the function $\bar{W}_{pq}(\alpha, \beta)$, $\alpha, \beta \in Z_2 \times Z_2$ is defined as

$$\bar{W}_{p,q}(\alpha, \beta) = (-1)^{Q(\alpha,\beta)} g_{pq}(0, \alpha - \beta) \tag{2.6}$$

where

$$Q(\alpha, \beta) = \beta_1(\beta_1 - \alpha_1) + \beta_2(\beta_1 - \alpha_1 + \beta_2 - \alpha_2) \qquad \alpha, \beta \in Z_2 \times Z_2 \tag{2.7}$$

and the function $g_{pq}(0, \alpha)$ has the following form:

$$g_{pq}(0, \alpha) = \frac{\prod_{\beta=0}^{\alpha_1+\alpha_2-1} (h_3^+(p) h_3^-(q) - h_3^+(q) h_3^-(p)(-1)^{-\beta})}{\prod_{i=1}^{2} \prod_{\beta_i=0}^{\alpha_i-1} (h_i^+(p) h_i^-(q) - h_i^+(q) h_i^-(p)(-1)^{1+\beta_i})}. \tag{2.8}$$

We choose a normalization of $\bar{W}_{pq}(\alpha, \beta)$ as

$$\bar{W}_{pq}(0, 0) = 1. \tag{2.9}$$

Then it is easy to see that

$$\bar{W}_{pp}(\alpha, \beta) = \bar{\delta}_{\alpha,\beta} \tag{2.10}$$

where

$$\bar{\delta}_{\alpha,\beta} \equiv \begin{cases} 1 & \alpha = \beta \pmod 2 \\ 0 & \text{otherwise.} \end{cases} \tag{2.11}$$

The function $\bar{W}_{pq}(\alpha, \beta)$ satisfies the inversion relation

$$\sum_{\beta \in Z_2 \times Z_2} \bar{W}_{pq}(\alpha, \beta) \bar{W}_{qp}(\beta, \gamma) = \bar{\delta}_{\alpha,\gamma} \Phi_{pq} \tag{2.12}$$

where the inversion factor Φ_{pq} is given by

$$\Phi_{pq} = \frac{4E_{pq}}{D_{pq}} \tag{2.13}$$

and

$$E_{pq} = \prod_{i=1}^{3} h_i^+(p)h_i^-(q) + \prod_{i=1}^{3} h_i^-(p)h_i^+(q) \tag{2.14}$$

$$D_{pq} = \prod_{i=1}^{3}(h_i^+(p)h_i^-(q) + h_i^-(p)h_i^+(q)). \tag{2.15}$$

As was shown in [8] (see [9] for details) the Boltzmann weights $\bar{W}$ satisfy the 'star–star' relation which provide the integrability of the $sl(n)$ chiral Potts model. This relation appears as

$$\frac{\bar{W}_{p'p}(\delta, \alpha)}{\bar{W}_{p'p}(\gamma, \beta)} W_{qq'}^{pp'}(\alpha, \beta, \gamma, \delta) = \tilde{W}_{q'q}^{p'p}(\alpha, \beta, \gamma, \delta) \frac{\bar{W}_{q'q}(\alpha, \beta)}{\bar{W}_{q'q}(\delta, \gamma)} \tag{2.16}$$

where the two 'star' weights are defined as follows:

$$W_{qq'}^{pp'}(\alpha, \beta, \gamma, \delta) = \sum_{\sigma} \frac{\bar{W}_{pq}(\alpha, \sigma)\bar{W}_{p'q'}(\gamma, \sigma)\bar{W}_{q'p}(\sigma, \beta)}{\bar{W}_{p'q}(\delta, \sigma)} \tag{2.17}$$

$$\tilde{W}_{q'q}^{p'p}(\alpha, \beta, \gamma, \delta) = \sum_{\sigma} \frac{\bar{W}_{pq}(\sigma, \gamma)\bar{W}_{p'q'}(\sigma, \alpha)\bar{W}_{q'p}(\delta, \sigma)}{\bar{W}_{p'q}(\sigma, \beta)}. \tag{2.18}$$

The objects defined in (2.17) and (2.18) are the 'star' weights. As was mentioned above, these weights correspond to the three-layer case of the Zamolodchikov model. To be exact for the general case of the rapidity variables $h_i^{\pm}(p)$ satisfying (2.1) the corresponding Zamolodchikov model is inhomogeneous in the third direction. In fact, we are mainly interested in the homogeneous case

$$h_i^+(p) = 1 \qquad h_i^-(p) = p. \tag{2.19}$$

It is easy to see that the defining relations (2.4) are trivially satisfied. Therefore, we do not need to work with the high-genus curve Γ.

As it was pointed out in [10] the rapidity variables can be parametrized in terms of the spherical angles and excesses θ_1, θ_2, a_3

$$\frac{q'}{q} = -\mathrm{i}\tan\frac{\theta_2}{2} \qquad \frac{p}{p'} = \mathrm{i}\tan\frac{\theta_1}{2} \qquad \frac{p}{q} = \mathrm{e}^{-\mathrm{i}\frac{a_3}{2}}\sqrt{\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}}. \tag{2.20}$$

3. Transfer matrices

Here we use slightly different definitions of the transfer matrices compared with [18]:

$$T(p; q, q')_{i_1,\ldots,i_N}^{j_1,\ldots,j_N} = \prod_{k=1}^{N} \frac{\bar{W}_{pq}(i_k, j_k)\bar{W}_{q'p}(j_k, i_{k+1})}{\bar{W}_{q'q}(i_k, i_{k+1})} \tag{3.1}$$

$$\bar{T}(p; q, q')_{i_1,\ldots,i_N}^{j_1,\ldots,j_N} = \prod_{k=1}^{N} \frac{\bar{W}_{q'q}(j_k, j_{k+1})\bar{W}_{pq'}(j_{k+1}, i_k)}{\bar{W}_{pq}(j_k, i_k)} \tag{3.2}$$

which are shown on figures 4 and 5, where we imply the cyclic boundary conditions $i_{N+1} = i_1$ and $j_{N+1} = j_1$.

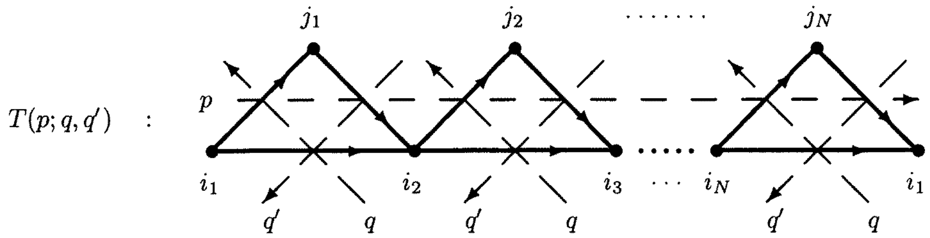

Figure 4.

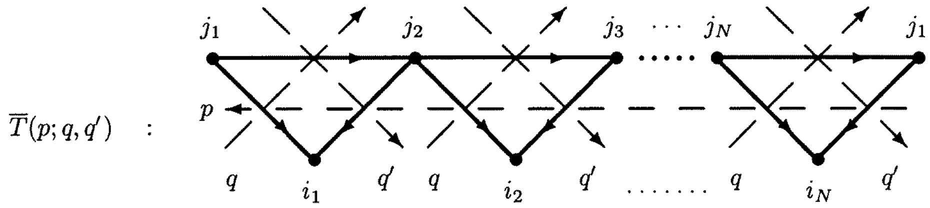

Figure 5.

We note that these definitions differ from the previous ones just by the diagonal equivalence transformation. Of course, it does not effect the partition function.

Below, we use more simple notations $T_p = T(p; q, q')$ and $\bar{T}_p = \bar{T}(p; q, q')$ assuming that the rapidities q and q' are fixed. Due to (2.16) these transfer matrices T_p and $\bar{T}_p$ commute. Namely, for two arbitrary rapidities p and p'

$$[T_p, T_{p'}] = [\bar{T}_p, \bar{T}_{p'}] = [T_p, \bar{T}_{p'}] = 0. \tag{3.3}$$

One can consider some limiting cases. From (2.6)–(2.9) we can conclude that if $q' \to p$ we have

$$T_p = X^{-1} \qquad \bar{T}_p = X \tag{3.4}$$

where X is the shift-operator:

$$X^{j_1 \ldots j_N}_{i_1 \ldots i_N} = \prod_{k=1}^{N} \delta_{i_k, j_{k+1}}. \tag{3.5}$$

If $q \to p$ then

$$T_p = I \tag{3.6}$$

while $\bar{T}$ has the singular matrix elements.

4. Functional relations

Further, we only consider the case of the homogeneous three-layer Zamolodchikov model. Due to the commutation relations (3.3) we can diagonalize the transfer matrices T_p and $\bar{T}_p$ simultaneously.

Let us denote eigenvalues of T_p and $\bar{T}_p$ by $t(p)$ and $\bar{t}(p)$ where we omit a dependence on q and q'.

In the appendix we outline the proof of the following pair of functional relations

$$\begin{aligned}\bar{t}(p)t(p)\bar{t}(-p)t(-\omega p)\\ = \phi_0^N \bar{t}(p)t(-\omega p) + \phi_1^N \bar{t}(-p)t(-\omega p) + \phi_2^N \bar{t}(p)t(\omega p) + \phi_3^N \bar{t}(-p)t(\omega p)\end{aligned} \tag{4.1}$$

and

$$\bar{t}(-\omega p)t(-p)\bar{t}(p)t(p) \\ = \phi_0'^N \bar{t}(-\omega p)t(p) + \phi_1'^N \bar{t}(-\omega p)t(-p) + \phi_2'^N \bar{t}(\omega p)t(p) + \phi_3'^N \bar{t}(\omega p)t(-p) \quad (4.2)$$

where

$$\phi_0 = 4\frac{(p+\omega q)(p+\omega^{-1}q)}{(p+q)^2} \qquad \phi_1 = 4\frac{(p+\omega q')(p+\omega^{-1}q')}{(p+q')^2} \quad (4.3)$$

$$\begin{aligned} \phi_2 &= 4\frac{(p-q)(p+\omega^{-1}q)^2(p+\omega^{-1}q')(p-\omega q')}{(p-\omega^2 q)(p+q)^2(p+q')(p-\omega^{-1}q')} \\ \phi_3 &= 4\frac{(p-q')(p+\omega^{-1}q')^2(p+\omega^{-1}q)(p-\omega q)}{(p-\omega^2 q')(p+q')^2(p+q)(p-\omega^{-1}q)} \end{aligned} \quad (4.4)$$

and ϕ_i' can be obtained from ϕ_i by the substitution $q \to -q$. Here ω is the root of unity of power three

$$\omega = \mathrm{e}^{\frac{\pm 2\pi \mathrm{i}}{3}}.$$

From the limiting cases (3.4) and (3.6) we have some initial data

$$t(p;q,p) = \Omega \qquad \bar{t}(p;q,p) = \Omega^{-1} \qquad t(p;p,q') = 1 \quad (4.5)$$

where Ω is some root of unity of power N:

$$\Omega^N = 1. \quad (4.6)$$

From (4.1) and (4.2) one can see that the pair of functions t' and $\bar{t}'$

$$t'(p;q,q') = \bar{t}(p;-q,q') \qquad \bar{t}'(p;q,q') = t(p;-q,q') \quad (4.7)$$

satisfy the same relations (4.1) and (4.2). However, it is not true that $\bar{t}(p;q,q') = t(p;-q,q')$ for all eigenvalues. The transformation (4.7) also interchanges the eigenvectors of the transfer matrices which belong to the same symmetry sector.

The analysis of the eigenvalues $t(p)$ and $\bar{t}(p)$ shows that it is convenient to extract some 'kinematic' multipliers:

$$t(p) = \frac{2^N}{(p+q)^N(p+q')^N}s(p) \qquad \bar{t}(p) = \frac{2^N}{(p-q)^N(p+q')^N}\bar{s}(p) \quad (4.8)$$

where $s(p)$ and $\bar{s}(p)$ are the polynomials of the degree n in the variable p. So far, we have no proof that the degrees of $s(p)$ and $\bar{s}(p)$ should be the same. Therefore, we accept it as a conjecture.

Substituting the definitions (4.8) into (4.1) and (4.2) we obtain the functional relations for $s(p)$ and $\bar{s}(p)$:

$$\bar{s}(p)s(p)\bar{s}(-p)s(-\omega p) \\ = \lambda_0^N \bar{s}(p)s(-\omega p) + \lambda_1^N \bar{s}(-p)s(-\omega p) + \lambda_2^N \bar{s}(p)s(\omega p) + \lambda_3^N \bar{s}(-p)s(\omega p) \quad (4.9)$$

and

$$\bar{s}(-\omega p)s(-p)\bar{s}(p)s(p) \\ = \lambda_0'^N \bar{s}(-\omega p)s(p) + \lambda_1'^N \bar{s}(-\omega p)s(-p) + \lambda_2'^N \bar{s}(\omega p)s(p) + \lambda_3'^N \bar{s}(\omega p)s(-p) \quad (4.10)$$

where

$$\begin{aligned} \lambda_0 &= (p+\omega q)(p+\omega^{-1}q)(p+q')(p-q') \\ \lambda_1 &= (p+\omega q')(p+\omega^{-1}q')(p+q)(p-q) \\ \lambda_2 &= (p-q)(p+\omega^{-1}q)(p-\omega q')(p-q') \\ \lambda_3 &= (p-q')(p+\omega^{-1}q')(p-\omega q)(p-q) \end{aligned}$$

and λ_i' can be obtained from λ_i by the substitution $q \to -q$.

5. Bethe ansatz equations

To construct the Bethe ansatz we consider zeros of the polynomials $s(p)$ and $\bar{s}(p)$:

$$s(p) = a_n(q, q') \prod_{i=1}^{n} (p - p_i) \qquad \bar{s}(p) = \bar{a}_n(q, q') \prod_{i=1}^{n} (p - \bar{p}_i) \tag{5.1}$$

where the power n takes only two possible values $2N$ and $2N - 1$. The functions a_n and $\bar{a}_n$ should be compatible with the initial conditions (4.5). Unfortunately, it is not easy to calculate them explicitly but their product looks very simple:

$$a_{2N}(q, q')\bar{a}_{2N}(q, q') = 4 \qquad a_{2N-1}(q, q')\bar{a}_{2N-1}(q, q') = N(q'^2 - q^2). \tag{5.2}$$

Now we can set p to be some zero of the LHS of (4.9) and consider the equations which follow from the RHS. In fact, we have four possibilities to do this:

$$p \to \bar{p}_i \qquad p \to -\bar{p}_i \qquad p \to -\omega^{-1} p_i \qquad p \to p_i. \tag{5.3}$$

It is not difficult to obtain that the first three possibilities give us two different sets of Bethe ansatz equations:

$$\frac{f(p_i, \omega^{\pm 1}, -q)^N}{f(p_i, \omega^{\pm 1}, -q')^N} = (-1)^{n-1} \prod_{j=1}^{n} \frac{p_i + \omega^{\mp 1} \bar{p}_j}{p_i - \omega^{\mp 1} \bar{p}_j} \tag{5.4}$$

and

$$\frac{f(\bar{p}_i, \omega^{\pm 1}, q)^N}{f(\bar{p}_i, \omega^{\pm 1}, -q')^N} = (-1)^{n-1} \prod_{j=1}^{n} \frac{\bar{p}_i + \omega^{\mp 1} p_j}{\bar{p}_i - \omega^{\mp 1} p_j} \tag{5.5}$$

where

$$f(p, x, q) = \frac{p - xq}{p + q}. \tag{5.6}$$

The fourth possibility in (5.3) gives some complicated compatibility conditions for the solution to the Bethe ansatz equations (5.4) and (5.5). Of course, p_i and $\bar{p}_i$ are the functions of q and q'. A similar consideration of the second functional relation leads to the same Bethe ansatz equations (5.4) and (5.5). It is obvious that $s(p)$ and $\bar{s}(p)$ are homogeneous in p, q, q'. So let

$$q = 1 \qquad p = \mathrm{i}x \qquad q' = \mathrm{i}y \tag{5.7}$$

where x, y are *real*.

Conjecture.

$$\bar{s}(x, y) = s^*(x, y). \tag{5.8}$$

We checked this numerically for $N = 2, 3$. Let us set

$$p_i = i r_i(y) \qquad \bar{p}_i = i \bar{r}_i(y). \tag{5.9}$$

Then

$$s(p) = a(y) i^n \prod_{i=1}^{n} (x - r_i(y)) \qquad \bar{s}(p) = \bar{a}(y) i^n \prod_{i=1}^{n} (x - \bar{r}_i(y)). \tag{5.10}$$

From (5.4) we obtain

$$\bar{a}(y) = (-1)^n a^*(y) \qquad \bar{r}_i(y) = r_i^*(y) \qquad i = 1, \ldots, n. \tag{5.11}$$

It is easy to see that (5.5) can be obtained from (5.4) by a complex conjugation. Somehow, this is the ‘proof’ of the conjecture (5.8). Then we have

$$\left[\frac{(r_i - y)(i r_i + \omega^\epsilon)}{(r_i + \omega^\epsilon y)(i r_i - 1)} \right]^N = (-1)^{n-1} \prod_{j=1}^{n} \frac{r_i + \omega^{-\epsilon} r_j^*}{r_i - \omega^{-\epsilon} r_j^*} \qquad \epsilon = \pm 1. \tag{5.12}$$

One can obtain, from (5.12), the set of equations on absolute values and phases of r_i.

6. Discussion

In this paper we have only presented the Bethe ansatz equations. We shall give the detailed analysis of these equations elsewhere. The technique we use here is in the spirit of the Baxter's Q-matrix method [12]. The role of the Q-matrices is played by the one-layer transfer matrices. It corresponds to the result obtained by Bazhanov and Stroganov in [11] for the chiral Potts model. We think that the algebraic Bethe ansatz technique can also be developed. However, there are some problems, such as an appropriate choice of the reference state, which are presently beyond our understanding.

We should note that the functional relations we have derived here and those which were obtained in [18] can be considered together. Perhaps the combination of all these relations could give more information about the eigenvalues $t(p)$ and $\bar{t}(p)$.

We hope that the result obtained by Baxter in [7] for the partition function of the Zamolodchikov model on the lattice $\infty \times \infty \times 3$ can be reproduced in the thermodynamic limit of the Bethe ansatz equations (5.4) and (5.5). We also hope that a standard programme of a study of the excitations and finite size corrections† can be performed.

We think that the technique described in the appendix can be generalized to the $sl(n)$ case. In principle, a general procedure seems to be more or less clear. However, the technical difficulties could be, of course, much more serious.

Acknowledgments

The authors would like to thank Murray Batchelor, Rodney J Baxter, Vladimir Bazhanov, Rainald Flume, Vladislav Fridkin, Günter von Gehlen, Jean-Michel Maillet and Vladimir Rittenberg for stimulating discussions and suggestions. HEB would also like to thank R Flume for his kind hospitality in the Physical Institute of Bonn University. This research (VVM) has been supported by the Australian Research Council and (HEB) by the Alexander von Humboldt Foundation.

Appendix A.

Here we outline the derivation of (4.1). The second relation (4.2) can be obtained in a similar way. In fact, we derive it for a general case of the inhomogeneous Zamolodchikov model. Our key relation for the transfer matrices T_p and $\bar{T}_p$ we would like to obtain looks as follows:

$$\bar{T}_p T_p \bar{T}_{\tau(p)} T_{p^\star} = \Phi_0^N \bar{T}_p T_{p^\star} + \Phi_1^N \bar{T}_{\tau(p)} T_{p^\star} + \Phi_2^N \bar{T}_p T_{\tau(p^\star)} + \Phi_3^N \bar{T}_{\tau(p)} T_{\tau(p^\star)} \tag{A.1}$$

where $p^\star$ is one of two nontrivial solutions of the equation

$$\frac{H_p^+}{H_p^-} = -\frac{H_{p^\star}^+}{H_{p^\star}^-} \tag{A.2}$$

where

$$H_p^\pm = \prod_{i=1}^{3} h_i^\pm(p) \qquad u_{pq} = \frac{\Delta_{pq}}{D_{pq}} \frac{D_{\tau(p^\star)q}}{\Delta_{\tau(p^\star)q}} \qquad v_{pq} = \frac{E_{\tau(p)q}}{\Delta_{\tau(p)q}} \frac{\Delta_{p^\star q}}{D_{p^\star q}} \tag{A.3}$$

and

$$\Phi_0 = 4\frac{E_{pq}}{D_{pq}} \qquad \Phi_1 = 4\frac{E_{pq'}}{D_{pq'}} \qquad \Phi_2 = 4u_{pq}v_{pq'} \qquad \Phi_3 = 4v_{pq}u_{pq'}. \tag{A.4}$$

† See, for example, [13] and references therein.

The function E, D and Δ are given by the formulae (2.14), (2.15) and (2.4) respectively.

In fact, in our previous paper [18] we made the first step. Namely, we expressed the matrix product $T_p\bar{T}_{\tau(p)}$ as a sum of two terms†. The first one corresponds to the first term in the RHS of (A.1). The second term was written in terms of some L-operators. When the vertical rapidities q and q' coincide this L-operator corresponds to the second fundamental representation $\bar{3}$ of the quantum $sl(3)$ algebra. Therefore, we shall denote it as $\bar{L}$.

Now we have to perform the next step. Namely, we consider the matrix product

$$(\bar{T}_p\bar{L}_p)^{\{\alpha\}}_{\{\gamma\}} = \mathrm{Tr}\prod_{i=1}^{N} B^{\alpha_i}_{\gamma_i,\gamma_{i+1}}(q', q; p) \tag{A.5}$$

where

$$[B^{\alpha}_{\gamma,\delta}(q', q; p)]_{i,j} = \sum_{\beta} \frac{\bar{W}_{pq'}(\beta,\gamma)}{\bar{W}_{pq}(\beta,\delta)}\bar{L}_{ij}(\beta,\alpha). \tag{A.6}$$

$\bar{L}$ is given by

$$\bar{L}_{i,j}(\beta,\alpha) = \sum_{n,m} C(i,n)\bar{W}_{\tau(p)q'}(\alpha,n)\bar{W}_{q'p}(n,\beta)\frac{\bar{W}_{pq}(\beta,m)}{\bar{W}_{\tau(p)q}(\alpha,m)}C(j,m) \tag{A.7}$$

and all indices $\alpha,\beta,\gamma,\delta,n,m$ are two-component vectors taking one of four possible states $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$, $i,j = 1,2,3$,

$$C(2k_1+k_2,n) = \tfrac{1}{2}(-1)^{k_1n_1+k_2n_2}. \tag{A.8}$$

Inserting the identity matrices 3×3 between each pair of B in the RHS of (A.5)

$$I = \sum_{i=1}^{3}\phi_R(i,\alpha)\times\phi_L(i,\alpha) \tag{A.9}$$

where

$$\phi_L(1,\alpha) = (1,0,0) \qquad \phi_L(2,\alpha) = (-(-1)^{\alpha_1+\alpha_2},1,0) \qquad \phi_L(3,\alpha) = (-(-1)^{\alpha_1},0,1) \tag{A.10}$$

$$\phi_R(1,\alpha) = (1,(-1)^{\alpha_1+\alpha_2},(-1)^{\alpha_1}) \qquad \phi_R(2,\alpha) = (0,1,0) \qquad \phi_R(3,\alpha) = (0,0,1) \tag{A.11}$$

one can check that the transformed matrices

$$[\tilde{B}^{\alpha}_{\gamma,\delta}(q', q; p)]_{ij} = \phi_L(i,\gamma)B^{\alpha}_{\gamma,\delta}(q', q; p)\phi_R(j,\delta) \tag{A.12}$$

satisfy the following property:

$$[\tilde{B}^{\alpha}_{\gamma,\delta}(q', q; p)]_{21} = [\tilde{B}^{\alpha}_{\gamma,\delta}(q', q; p)]_{31} = 0 \tag{A.13}$$

for all possible values of indices α,γ,δ.

Therefore, we have a decomposition $1+2$. It is not difficult to check that

$$[\tilde{B}^{\alpha}_{\gamma,\delta}(q', q; p)]_{11} = \Phi_{pq'}(-1)^{\gamma_2+\delta_2}\frac{\bar{W}_{\tau(p)q'}(\alpha,\gamma)}{\bar{W}_{\tau(p)q}(\alpha,\delta)} \tag{A.14}$$

where Φ_{pq} is defined in (2.13). In the RHS of formula (A.14) we can recognize the 'building block' of the transfer matrix $\bar{T}_{\tau(p)}$. So, after taking the product and trace as in the RHS of (A.5) we obtain the second term in (A.1).

† We considered two cases $\lambda = 0, 1$ corresponding to two automorhisms τ^{λ} (see formula (3.2) of [18]). Here we consider only the case $\lambda = 1$. A consideration of the case $\lambda = 0$ gives the same result.

Now let us define 2×2 matrices with elements

$$[\hat{B}^{\alpha}_{\gamma,\delta}(q', q; p)]_{ij} = [\tilde{B}^{\alpha}_{\gamma,\delta}(q', q; p)]_{i+1,j+1} \qquad i, j = 1, 2. \tag{A.15}$$

The matrices $\hat{B}^{\alpha}_{\gamma,\delta}(q', q; p)$ have the form of the following matrix product:

$$\hat{B}^{\alpha}_{\gamma,\delta}(q', q; p) = V_{pq'}(\alpha, \gamma) U_{pq}(\alpha, \delta) \tag{A.16}$$

where

$$[U_{pq}(\alpha, \delta)]_{ij} = \sum_{n,m} \chi_L(i; m, \alpha) \frac{\bar{W}_{pq}(m, n)}{\bar{W}_{pq}(m, \delta) \bar{W}_{\tau(p)q}(\alpha, n)} \chi_R(j; n, \delta) \tag{A.17}$$

and

$$[V_{pq}(\alpha, \delta)]_{ij} = \sum_{n,m} \chi_L(i; m, \delta) \bar{W}_{\tau(p)q}(\alpha, m) \bar{W}_{qp}(m, n) \bar{W}_{pq}(n, \delta) \chi_R(j; n, \alpha). \tag{A.18}$$

Here we use the following notations:

$$\chi_{L(R)}(i; m, \alpha) = \sum_{k=1}^{3} C(k, m) [\phi_{L(R)}(i+1; \alpha)]_k \qquad i = 1, 2. \tag{A.19}$$

It is interesting to note that U_{pq} and V_{pq} satisfy the property which is similar to that for $\tilde{B}$ given by (A.13)

$$[U_{pq}(\alpha, \delta)]_{i0} = [V_{pq}(\alpha, \delta)]_{i0} = 0. \tag{A.20}$$

In addition, we have

$$[U_{pq}(\alpha, \delta)]_{00} = -\frac{\Phi_{pq}}{4} (-1)^{\alpha_2+\delta_2} \bar{W}_{q\tau(p)}(\delta, \alpha) \tag{A.21}$$

$$[V_{pq}(\alpha, \delta)]_{00} = -\frac{\Phi_{pq}}{4} (-1)^{\alpha_2+\delta_2} \bar{W}_{\tau(p)q}(\alpha, \delta) \tag{A.22}$$

and Φ_{pq} is given by (2.13).

Using the definitions (A.17) and (A.18) we obtain

$$U_{pq}(\alpha, \delta) = (-1)^{\alpha_1+\alpha_2} \eta_{pq}(\alpha_1, \alpha_2; \delta_1, \delta_2) \begin{pmatrix} -\frac{1}{\gamma_2(p,q)} & z_{12}(p, q; \alpha, \delta) \\ -z_{32}(p, q; \alpha, \delta) & 1 \end{pmatrix} \tag{A.23}$$

$$V_{pq}(\alpha, \delta) = (-1)^{\alpha_1+\delta_2} \eta_{pq}(\alpha_1, \alpha_2; \delta_1, \delta_2) \begin{pmatrix} 1 & -z_{12}(p, q; \alpha, \delta) \\ z_{32}(p, q; \alpha, \delta) & -\frac{1}{\gamma_2(p,q)} \end{pmatrix} \tag{A.24}$$

where

$$\gamma_i(p, q) = -\frac{h_i^+(p) h_i^-(q)}{h_i^-(p) h_i^+(q)} \qquad \eta_{pq}(\alpha_1, \alpha_2; \delta_1, \delta_2) = -\frac{2\Delta_{pq}}{D_{pq}} \frac{h_2^+(p) h_2^-(q)}{\bar{W}_{pq}(\alpha_1+1, \alpha_2; \delta_1, \delta_2)} \tag{A.25}$$

$$z_{12}(p, q; \alpha, \delta) = (-1)^{\alpha_2} \frac{\gamma_1(p, q)\gamma_2(p, q) - (-1)^{\alpha_1+\delta_1+\alpha_2+\delta_2}}{(\gamma_1(p, q) - (-1)^{\alpha_1+\delta_1})\gamma_2(p, q)} \tag{A.26}$$

$$z_{32}(p, q; \alpha, \delta) = (-1)^{\delta_2} \frac{\gamma_3(p, q)\gamma_2(p, q) - (-1)^{\alpha_1+\delta_1}}{(\gamma_3(p, q) - (-1)^{\alpha_1+\delta_1+\alpha_2+\delta_2})\gamma_2(p, q)}. \tag{A.27}$$

It is easy to see from (A.23) and (A.24) that the matrix U_{pq} is connected with V_{pq} by the matrix inversion up to some coefficient:

$$V_{pq} U_{pq} = \frac{(\gamma_1(p, q)\gamma_2(p, q)\gamma_3(p, q) - 1)(\gamma_2(p, q) - (-1)^{\alpha_2+\delta_2})\eta_{p,q}(\alpha_1, \alpha_2; \delta_1, \delta_2)^2}{(\gamma_1(p, q) - (-1)^{\alpha_1+\delta_1})(\gamma_3(p, q) - (-1)^{\alpha_1+\delta_1+\alpha_2+\delta_2})\gamma_2(p, q)^2}. \tag{A.28}$$

The important fact is a degeneration of these matrices which occurs when

$$\gamma_1(p,q)\gamma_2(p,q)\gamma_3(p,q) = 1. \tag{A.29}$$

Using the 'star–star' relation for the Boltzmann weights $\bar{W}$ and property (A.20) we can prove that the matrices U and V should satisfy the following important relation:

$$\sum_{\alpha} V_{pq'}(\alpha,\gamma)U_{pq}(\alpha,\delta)V_{pp'}(\delta',\delta)\bar{W}_{q'p'}(\gamma',\alpha)\bar{W}_{p'q}(\alpha,\delta') \\ = \frac{\bar{W}_{q'q}(\gamma',\delta')}{\bar{W}_{q'q}(\gamma,\delta)} \sum_{\beta} \bar{W}_{p'q}(\gamma,\beta)\bar{W}_{q'p'}(\beta,\delta)V_{pp'}(\gamma',\gamma)U_{pq}(\gamma',\beta)V_{pq'}(\delta',\beta). \tag{A.30}$$

From this relation we can deduce that choosing the rapidity variable p' in such a way that all matrices $V_{pp'}$ are degenerate we get the decomposition of the matrices with the following elements:

$$[D^{\gamma',\delta'}_{\gamma,\delta}(q',q;\,p,p')]_{ij} = \sum_{\alpha} [V_{pq'}(\alpha,\gamma)U_{pq}(\alpha,\delta)]_{ij}\bar{W}_{q'p'}(\gamma',\alpha)\bar{W}_{p'q}(\alpha,\delta'). \tag{A.31}$$

It means that we can reduce the matrices $D^{\gamma',\delta'}_{\gamma,\delta}(q',q;\,p,p')$ by the quasi-equivalence transformation to the upper-triangular form. This technique is rather similar to that which was used by Baxter for a derivation of the Q-matrix equation for the six-vertex and eight-vertex models [12].

So, first we should choose the point on the curve p' to provide the degeneration of matrices $V_{p,p'}$. Therefore, we should fulfil the condition (A.29) for the pair (p,p'):

$$\gamma_1(p,p')\gamma_2(p,p')\gamma_3(p,p') = 1. \tag{A.32}$$

This equation has three solutions (up to some choice of signs). One of them

$$\Delta_{pp'} = 0 \tag{A.33}$$

corresponds to the automorphism τ:

$$p' = \tau(p). \tag{A.34}$$

Two another solutions can be obtained by taking the second power of (A.32) and using (2.1). In this way we arrive at the quadratic equation for the coordinates of p' with coefficients depending on coordinates of p. Let us denote its roots as

$$p_{\pm} = \tau_{\pm}(p). \tag{A.35}$$

Let us choose one of these solutions, for example,

$$p^{\star} = p_{+} \tag{A.36}$$

and set the point p' in the formulae above to be $p^{\star}$.

It is easy to conclude from (A.24) that up to some factor the matrices $V_{pp^{\star}}$ are proportional to

$$V_{pp^{\star}}(\alpha,\delta) \sim \begin{pmatrix} 1 & -(-1)^{\alpha_2}\frac{\gamma_1^{\star}\gamma_2^{\star}-(-1)^{\alpha_1+\delta_1+\alpha_2+\delta_2}}{(\gamma_1^{\star}-(-1)^{\alpha_1+\delta_1})\gamma_2^{\star}} \\ \frac{(-1)^{\alpha_2}(\gamma_1^{\star}-(-1)^{\alpha_1+\delta_1})}{\gamma_1^{\star}\gamma_2^{\star}-(-1)^{\alpha_1+\delta_1+\alpha_2+\delta_2}} & -\frac{1}{\gamma_2^{\star}} \end{pmatrix} \tag{A.37}$$

where

$$\gamma_i^{\star} = \gamma_i(p,p^{\star}). \tag{A.38}$$

So, the vectors which provide the decomposition 1 + 1 can be chosen as:

$$\zeta_L(1;\alpha,\delta)=(1,0)\qquad \zeta_L(2;\alpha,\delta)=\left(-(-1)^{\alpha_2}\frac{\gamma_1^\star\gamma_2^\star-(-1)^{\alpha_1+\delta_1+\alpha_2+\delta_2}}{\gamma_1^\star-(-1)^{\alpha_1+\delta_1}},1\right) \tag{A.39}$$

$$\zeta_R(2;\alpha,\delta)=(0,1)\qquad \zeta_R(1;\alpha,\delta)=\left(1,(-1)^{\alpha_2}\frac{\gamma_1^\star-(-1)^{\alpha_1+\delta_1}}{\gamma_1^\star\gamma_2^\star-(-1)^{\alpha_1+\delta_1+\alpha_2+\delta_2}}\right) \tag{A.40}$$

which satisfy the natural condition

$$\sum_{i=1}^{2}\zeta_R(i;\alpha,\delta)\times\zeta_L(i;\alpha,\delta)=I. \tag{A.41}$$

Now let us consider the transformed matrices $\hat{D}_{\gamma,\delta}^{\gamma',\delta'}$:

$$[\hat{D}_{\gamma,\delta}^{\gamma',\delta'}(q',q;p)]_{ij}=\zeta_L(i;\gamma',\gamma)D_{\gamma,\delta}^{\gamma',\delta'}(q',q;p,p^\star)\zeta_R(j;\delta',\delta) \tag{A.42}$$

where $i,j=1,2$.

One can check the decomposition property

$$[\hat{D}_{\gamma,\delta}^{\gamma',\delta'}(q',q;p)]_{21}=0. \tag{A.43}$$

Now we should study diagonal elements of these matrices $[\hat{D}_{\gamma,\delta}^{\gamma',\delta'}(q',q;p)]_{ii}$, $i=1,2$.

It can be checked that the following expressions for $\hat{D}_{ii}$ are valid:

$$[\hat{D}_{\gamma,\delta}^{\gamma',\delta'}(q',q;p)]_{ii}=\Lambda_i(q',q;p)\frac{A_i(\delta',\delta)}{A_i(\gamma',\gamma)}\hat{W}_{q'q}^{p_i\tau(p^\star)}(\gamma,\delta,\gamma',\delta') \tag{A.44}$$

where $\hat{W}_{q'q}^{p_i\tau(p^\star)}(\gamma,\delta,\gamma',\delta')$ is given by (2.18) and

$$p_1=p\qquad p_2=\tau(p). \tag{A.45}$$

For the scalar functions Λ_i we have

$$\Lambda_1(q',q;p)=4v_{q'p}u_{qp}\qquad \Lambda_2(q',q;p)=4u_{q'p}v_{qp} \tag{A.46}$$

and the functions u and v are given by (A.3). The gauge matrices A_i are given by:

$$A_1=\begin{pmatrix}1&a_1&a_1&1\\a_1&1&1&a_1\\-a_1&-1&-1&-a_1\\-1&-a_1&-a_1&-1\end{pmatrix}\qquad A_2=\begin{pmatrix}1&a_2&-a_2&-1\\-a_2&-1&1&a_2\\a_2&1&-1&-a_2\\-1&-a_2&a_2&1\end{pmatrix} \tag{A.47}$$

where

$$a_1=\frac{h_3^-(p)h_3^+(p^\star)+h_3^+(p)h_3^-(p^\star)}{h_3^-(p)h_3(p^\star)-h_3^+(p)h_3^-(p^\star)}\qquad a_2=-1/a_1. \tag{A.48}$$

So, we have succeeded in reducing the four-index objects $[\hat{D}_{\gamma,\delta}^{\gamma',\delta'}(q',q;p)]_{ii}$ to the original 'star' form. It is not difficult to observe that after taking the product and trace we obtain the last two terms in (A.1). Using commutation relations (3.3) we can simultaneously diagonalize the transfer matrices and get the functional relation for the eigenvalues. Taking into account that for the homogeneous case of the Zamolodchikov model the automorphism τ acts just as negating of p:

$$\tau(p)=-p. \tag{A.49}$$

and rapidity $p^\star$ can be taken as

$$p^\star=-\omega p \tag{A.50}$$

we come to (4.1).

References

[1] Zamolodchikov A B 1980 *Zh. Eksp. Teor. Fiz.* **79** 641–64
Zamolodchikov A B 1980 *Sov. Phys.–JETP* **52** 325–36 (Engl. transl.)
[2] Zamolodchikov A B 1981 *Commun. Math. Phys.* **79** 489–505
[3] Bazhanov V V and Stroganov Yu G 1982 *Teor. Mat. Fiz.* **52** 105–13
Bazhanov V V and Stroganov Yu G 1982 *Theor. Math. Phys.* **138** 685–91 (Engl. transl.)
[4] Jaekel M T and Maillard J M 1982 *J. Stat. Phys.* A **15** 1309
[5] Baxter R J 1983 *Commun. Math. Phys.* **88** 185
[6] Bazhanov V V and Stroganov Yu G 1985 *Theor. Mat. Fiz.* **63** 417
Bazhanov V V and Stroganov Yu G 1985 *Theor. Math. Phys.* **63** 604 (Engl. transl.)
[7] Baxter R J 1986 *Physica* D **18** 321–47
[8] Kashaev R M 1991 Private communication
[9] Bazhanov V V and Baxter R J 1992 *J. Stat. Phys.* **69** 453–85
[10] Bazhanov V V and Baxter R J 1993 *J. Stat. Phys.* **71** 839
[11] Bazhanov V V and Stroganov Yu G 1990 *J. Stat. Phys.* **59** 799
[12] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)
[13] Korepin V E, Bogoliubov N M and Izergin A G 1993 *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge: Cambridge University Press)
[14] Bazhanov V V, Kashaev R M, Mangazeev V V and Stroganov Yu G 1991 *Commun. Math. Phys.* **138** 393–408
[15] Date E, Jimbo M, Miki K and Miwa T 1991 *Commun. Math. Phys.* **137** 133
[16] Kashaev R M, Mangazeev V V and Nakanishi T 1991 *Nucl. Phys.* B **362** 563
[17] Baxter R J and Quispel G R W 1990 *J. Stat. Phys.* **58** 411–30
[18] Boos H E and Mangazeev V V 1999 *J. Phys. A: Math. Gen.* **32** 3041–54